Submitted exclusively to the London Mathematical Society
DOI: 10.1112/S0000000000000000

ORBIT-HOMOGENEITY IN PERMUTATION GROUPS

PETER J. CAMERON AND ALEXANDER W. DENT

Abstract

This paper introduces the concept of orbit-homogeneity of permutation groups: a group G is orbit-t-homogeneous if two sets of cardinality t lie in the same orbit of G whenever their intersections with each G-orbit have the same cardinality. For transitive groups, this coincides with the usual notion of t-homogeneity. This concept is also compatible with the idea of partition transitivity introduced by Martin and Sagan.

Further, this paper shows that any group generated by orbit-t-homogeneous subgroups is orbit-t-homogeneous, and that the condition becomes stronger as t increases up to $\lfloor n/2 \rfloor$, where n is the degree. So any group G has a unique maximal orbit-t-homogeneous subgroup $\Omega_t(G)$, and $\Omega_t(G) \leq \Omega_{t-1}(G)$.

Some structural results for orbit-t-homogeneous groups and a number of examples are also given.

A permutation group G acting on a set V is said to be *t-homogeneous* if it acts transitively on the set of t-element subsets of V. Informally, this means that all t-element subsets of V are "alike" with respect to the action of G. If the action of G is intransitive, it cannot be t-homogeneous, since the intersections of different t-subsets with orbits of G may be different. We define a more general condition to cover this situation: we say that G is orbit-t-homogeneous on V if two t-sets which meet each orbit in the same number of points are equivalent under the action of G. We give a similar extension of the notion of partition transitivity introduced by Martin and Sagan [**8**].

As a result of the classification of the finite simple groups [**4**], all t-homogeneous permutation groups G on sets V with $1 < t < |V| - 1$ are known. (We may assume without loss that $t \leq |V|/2$. Without the classification it can be shown that such a group is, with certain known exceptions, always t-transitive, see [**5**, **6**, **7**]; and the list of the t-transitive groups, which can be found in [**1**], follows from the classification.) For our more general concept, the determination of orbit-t-homogeneous groups is not complete, but we give a number of results in this direction.

A permutation group G acting on a set V is said to be *orbit-t-homogeneous*, or *t-homogeneous with respect to its orbit decomposition*, if whenever S_1 and S_2 are t-subsets of V satisfying $|S_1 \cap \Delta| = |S_2 \cap \Delta|$ for every G-orbit Δ, there exists $g \in G$ with $S_1 g = S_2$. Thus, a group which is t-homogeneous in the usual sense is orbit-t-homogeneous; every group is orbit-1-homogeneous; and the trivial group is orbit-t-homogeneous for every t. Furthermore, a group is orbit-2-homogeneous if and only if it is 2-homogeneous on each orbit and, for every $\alpha \in V$, the point stabiliser G_α acts transitively on each orbit not containing α. It is also clear that a group of degree n is orbit-t-homogeneous if and only if it is orbit-$(n-t)$-homogeneous; so, in these cases, we may assume $t \leq n/2$ without loss of generality.

2000 *Mathematics Subject Classification* 20B10.

If two sets S_1 and S_2 are subsets of V satisfying $|S_1 \cap \Delta| = |S_2 \cap \Delta|$ for every G-orbit Δ then S_1 and S_2 are said to have the same structure with respect to G (or just to have the same structure if the group is obvious).

Theorem 4.3.4 of [**3**] is the following:

THEOREM 1. *If G and H are orbit-t-homogeneous on V, then so is $\langle GH \rangle$.*

Young extended the concept of homogeneous groups by investigating the relationship between permutation groups and partitions [**9**]. An ordered partition of V, $P = (P_1, P_2, \ldots, P_k)$, is said to have shape

$$|P| = (|P_1|, |P_2|, \ldots, |P_k|)\,.$$

A group element $g \in G$ is said to map the partition P onto a partition $Q = (Q_1, Q_2, \ldots, Q_k)$ if $P_i g = Q_i$ for all i. Obviously, a pre-requisite for this is that P and Q have the same structure with respect to G, i.e. that P_i and Q_i have the same structure for all i. The permutation group G is said to be *orbit-λ-transitive* if, for any two partitions of V that have shape λ and the same structure, P and Q say, there exists some $g \in G$ that maps P to Q. A permutation group of degree n is orbit-t-homogeneous if and only if it is orbit-λ-transitive, where $\lambda = (n-t, t)$.

The following is a more general version of Theorem 1.

THEOREM 2. *If G and H are orbit-λ-transitive on a finite set V, then so is $\langle GH \rangle$.*

Proof outline. We begin by showing that it suffices to prove that there exists a $\sigma \in \langle GH \rangle$ that maps one partition to another when the two partitions differ in that two elements, x_1 and x_2, have "swapped" parts. We note that, since these two elements must lie in the same $\langle GH \rangle$-orbit, there must be a finite chain of elements $g_1 h_1 \ldots g_m h_m$ that map one to the other. We show that we can map one partition to the other if $m = 1$ by splitting the proof into three cases based on whether the intermediate point $y = x_1 g_1$ lies in the same part of the partition as x_1, x_2, or neither of these points. We extend these results by induction to cover all values of m using similar arguments. Therefore the theorem holds.

Proof. Let $P = (P_1, P_2, \ldots)$ and $Q = (Q_1, Q_2, \ldots)$ be finite partitions of the finite set V that have the same structure with respect to $\langle GH \rangle$ and have shape λ.

We say that a point $x \in V$ is "bad" if $x \in P_i$ but $x \notin Q_i$ for some integer i. Hence, the bad points are the points that need to be "moved" in order to map P to Q. Since V is finite, we may enumerate these bad points $x_1, ..., x_k$ and assume, without loss of generality, that $x_1 \in P_1 \setminus Q_1$. Since P has the same structure as Q with respect to $\langle GH \rangle$, there must exist a bad point $y \in P_i \cap Q_1$, for some $i \neq 1$, that is in the same orbit as x_1. Consider the partition $P^{(1)}$ given by:

(i) $P_1^{(1)} = (P_1 \setminus \{x_1\}) \cup \{y\}$,
(ii) $P_i^{(1)} = (P_i \setminus \{y\}) \cup \{x_1\}$,
(iii) $P_j^{(1)} = P_j$ for all $j \neq 1, i$.

The partitions P and $P^{(1)}$ differ only in that x_1 and y have swapped parts, but $P^{(1)}$ has at most $k-1$ bad points. Hence (by a simple induction), it is easy to see that there exists a chain of partitions $P = P^{(0)}, P^{(1)}, P^{(2)}, \ldots, P^{(l)} = Q$ with the same structure and shape such that the only difference between $P^{(j)}$ and $P^{(j+1)}$ is the swapping of two bad points.

Therefore, in order to prove the theorem it is sufficient to show that there exists $\sigma \in \langle GH \rangle$ such that $P\sigma = Q$ when

(i) $P_1 = S_1 \cup \{x_1\}$ and $P_2 = S_2 \cup \{x_2\}$,
(ii) $Q_1 = S_1 \cup \{x_2\}$ and $Q_2 = S_2 \cup \{x_1\}$, and
(iii) $P_j = Q_j$ for all $j > 2$,

for some distinct $x_1, x_2 \in V$ and $S_1, S_2 \subseteq V \setminus \{x_1, x_2\}$. Since P and Q have the same structure with respect to $\langle GH \rangle$, x_1 and x_2 must lie in the same $\langle GH \rangle$-orbit and so there exists an element $\sigma' = g_1 h_1 \dots g_m h_m$ such that $x_1 \sigma' = x_2$.

Suppose that $m = 1$ and let $y = x_1 g_1$. Note that x_1 and y lie in the same G-orbit and that y and x_2 lie in the same H-orbit. If $y = x_1$ then x_1 and x_2 lie in the same H-orbit and so, as H is orbit-λ-transitive, there exists an element $h \in H$ that maps P onto Q. If $y = x_2$ then x_1 and x_2 lie in the same G-orbit and so, as G is orbit-λ-transitive, there exists an element $g \in G$ that maps P onto Q. We therefore assume that $y \neq x_1, x_2$. We split the proof into three cases depending on whether $y \in P_1$, $y \in P_2$ or $y \in P_i$ for some $i \geq 3$.

Suppose that $y \in P_1$, i.e. $S_1 = S_1' \cup \{y\}$, and consider the partition $R = (R_1, R_2, \dots)$ where

$$R_1 = S_1' \cup \{x_1, x_2\}, \quad R_2 = S_2 \cup \{y\}, \quad R_i = P_i = Q_i \text{ for all } i > 2.$$

The partitions P and R have the same structure with respect to H and both have shape λ. Hence there exists $h \in H$ such that $Ph = R$. Similarly the partitions R and Q have the same structure with respect to G and so there exists $g \in G$ such that $Rg = Q$. Hence the result holds.

Suppose that $y \in P_2$, i.e. $S_2 = S_2' \cup \{y\}$, and consider the partition $R = (R_1, R_2, \dots)$ where

$$R_1 = S_1 \cup \{y\}, \quad R_2 = S_2' \cup \{x_1, x_2\}, \quad R_i = P_i = Q_i \text{ for all } i > 2.$$

The partitions P and R have the same structure with respect to G and both have shape λ. Hence there exists $g \in G$ such that $Pg = R$. Similarly the partitions R and Q have the same structure with respect to H and so there exists $h \in H$ such that $Rh = Q$. Hence the result holds.

If $y \notin P_1 \cup P_2$ then, without loss of generality, it can be assumed that $y \in P_3$, i.e. $P_3 = S_3 \cup \{y\}$ for some $S_3 \subseteq V$. Consider the partitions $R = (R_1, R_2, \dots)$ where

$$R_1 = S_1 \cup \{y\}, \quad R_2 = S_2 \cup \{x_2\}, \quad R_3 = S_3 \cup \{x_1\},$$
$$R_i = P_i = Q_i \text{ for all } i > 3,$$

and $T = (T_1, T_2, \dots)$ where

$$T_1 = S_1 \cup \{x_2\}, \quad T_2 = S_2 \cup \{y\}, \quad T_3 = S_3 \cup \{x_1\},$$
$$T_i = P_i = Q_i \text{ for all } i > 3.$$

Note that both partitions have shape λ. The partitions P and R have the same structure with respect to G, hence there exists $g \in G$ such that $Pg = R$. The partitions R and T have the same structure with respect to H, hence there exists $h \in H$ such that $Rh = T$. The partitions T and Q have the same structure with respect to G, hence there exists $g' \in G$ such that $Tg' = Q$. Hence the result holds when $m = 1$.

Assume, as induction hypothesis, that the theorem holds for a given value of m

and consider the case when $\sigma' = g_1 h_1 \ldots g_{m+1} h_{m+1}$. Let $y = x g_1 h_1 \ldots g_m h_m$. If $y = x_1$ or $y = x_2$ then the result is obvious, so we will assume that this is not the case.

Suppose that $y \in P_1$, i.e. $S_1 = S_1' \cup \{y\}$, and consider the partition $R = (R_1, R_2, \ldots)$ where

$$R_1 = S_1' \cup \{x_1, x_2\}, \quad R_2 = S_2 \cup \{y\}, \quad R_i = P_i = Q_i \text{ for all } i > 2.$$

Since there exists an element $g_{m+1} h_{m+1} \in \langle GH \rangle$ such that $y g_{m+1} h_{m+1} = x_2$ there must exist an element $\sigma_1 \in \langle GH \rangle$ such that $P\sigma_1 = R$. Similarly, since there exists an element $g_1 h_1 \ldots g_m h_m \in \langle GH \rangle$ such that $x_1 g_1 h_1 \ldots g_m h_m = y$ there must exist, by induction, an element $\sigma_2 \in \langle GH \rangle$ such that $R\sigma_2 = Q$. Hence $P\sigma_1\sigma_2 = Q$.

Suppose that $y \in P_2$, i.e. $S_2 = S_2' \cup \{y\}$, and consider the partition $R = (R_1, R_2, \ldots)$ where

$$R_1 = S_1 \cup \{y\}, \quad R_2 = S_2' \cup \{x_1, x_2\}, \quad R_i = P_i = Q_i \text{ for all } i > 2.$$

Since there exists an element $g_1 h_1 \ldots g_m h_m \in \langle GH \rangle$ such that $x_1 g_1 h_1 \ldots g_m h_m = y$ there must exist, by induction, an element $\sigma_1 \in \langle GH \rangle$ such that $P\sigma_1 = R$. Similarly, since there exists an element $g_{m+1} h_{m+1} \in \langle GH \rangle$ such that $y g_{m+1} h_{m+1} = x_2$ there must exist an element $\sigma_2 \in \langle GH \rangle$ such that $R\sigma_2 = Q$. Hence $P\sigma_1\sigma_2 = Q$.

If $y \notin P_1 \cup P_2$ then, without loss of generality, it can be assumed that $y \in P_3$, i.e. $P_3 = S_3 \cup \{y\}$ for some $S_3 \subseteq V$. Consider the partitions $R = (R_1, R_2, \ldots)$ where

$$R_1 = S_1 \cup \{y\}, \quad R_2 = S_2 \cup \{x_2\}, \quad R_3 = S_3 \cup \{x_1\},$$
$$R_i = P_i = Q_i \text{ for all } i > 3,$$

and $T = (T_1, T_2, \ldots)$ where

$$T_1 = S_1 \cup \{x_2\}, \quad T_2 = S_2 \cup \{y\}, \quad T_3 = S_3 \cup \{x_1\},$$
$$T_i = P_i = Q_i \text{ for all } i > 3.$$

Since there exists an element $g_1 h_1 \ldots g_m h_m \in \langle GH \rangle$ such that $x_1 g_1 h_1 \ldots g_m h_m = y$ there must exist, by induction, an element $\sigma_1 \in \langle GH \rangle$ such that $P\sigma_1 = R$. Similarly, since there exists an element $g_{m+1} h_{m+1} \in \langle GH \rangle$ such that $y g_{m+1} h_{m+1} = x_2$ there must exist an element $\sigma_2 \in \langle GH \rangle$ such that $R\sigma_2 = T$. Lastly, since there exists an element $g_1 h_1 \ldots g_m h_m \in \langle GH \rangle$ such that $x_1 g_1 h_1 \ldots g_m h_m = y$ there must exist, by induction, an element $\sigma_3 \in \langle GH \rangle$ such that $T\sigma_3 = Q$. (It may be supposed that such a σ_3 cannot be assumed to exist as $g_1 h_1 \ldots g_m h_m$ maps x_1 to y rather than mapping y to x_1, and the corresponding element of $\langle GH \rangle$ that maps y to x_1 is $e h_m^{-1} g_m^{-1} h_{m-1}^{-1} \ldots g_1^{-1} e \in \langle GH \rangle$ which is too long to apply the inductive assumption. However, since $g_1 h_1 \ldots g_m h_m$ maps x_1 to y, there exists a $\sigma \in \langle GH \rangle$ that maps $Q\sigma = T$. Therefore $\sigma_3 = \sigma^{-1}$ maps Q to T.) Hence $P\sigma_1\sigma_2\sigma_3 = Q$.

Therefore, the theorem holds for $\sigma' = g_1 h_1 \ldots g_{m+1} h_{m+1}$ and so, by induction, for all values of m. □

Hence any permutation group G on V has a unique subgroup $\Omega_\lambda(G)$ which is maximal with respect to being orbit-λ-transitive.

PROPOSITION 3. *For any permutation group G that acts on a finite set V, and any shape λ of V, the subgroup $\Omega_\lambda(G)$ is normal in G.*

Proof. Pick $g \in G$ and set $H = g\Omega_\lambda(G)g^{-1}$. Let P and Q be any two partitions of V with shape λ and the same structure with respect to H. Then Pg and Qg have shape λ and the same structure with respect to G, therefore there exists an element $\sigma \in \Omega_\lambda(G)$ such that $Pg\sigma = Qg$. Hence $P(g\sigma g^{-1}) = Q$ and so H is orbit-λ-transitive. Since $\Omega_\lambda(G)$ is a maximal orbit-λ-transitive subgroup of G, we have $H \leq \Omega_\lambda(G)$. However $|H| = |\Omega_\lambda(G)|$ and so $H = \Omega_\lambda(G)$. Therefore $\Omega_\lambda(G)$ is normal in G. □

If $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a shape of a partition of V then, without loss of generality, it can be assumed that

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k\,.$$

Furthermore, if $\mu = (\mu_1, \ldots, \mu_m)$ is the shape of another partition of V then a partial ordering can be defined where μ dominates λ, written $\lambda \trianglelefteq \mu$, if

$$\sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \mu_i$$

for all j (with the convention that $\lambda_i = 0$ for all $i > k$, and similarly for μ). Hence the set of shapes of V forms a lattice. A shape μ is said to *cover* a shape λ if

(1) λ and μ are distinct shapes such that μ dominates λ, and
(2) if ν is a shape with $\lambda \trianglelefteq \nu \trianglelefteq \mu$ then either $\lambda = \nu$ or $\mu = \nu$.

Hence, μ covers λ if it is immediately above λ in the shape lattice.

Proposition 4. *If μ covers λ in the shape lattice then there exists integers j, k such that $j < k$ and*

$$\mu_j = \lambda_j + 1, \quad \mu_k = \lambda_k - 1, \quad \mu_i = \lambda_i \text{ for } i \neq j, k.$$

Proof. Since μ dominates λ we must have that

$$\sum_{i=1}^{l} \lambda_i \leq \sum_{i=1}^{l} \mu_i$$

for all $l \in \mathbb{Z}$. Let j be the first value for which

$$\sum_{i=1}^{j} \lambda_i < \sum_{i=1}^{j} \mu_i\,,$$

i.e. the least integer such that $\lambda_i = \mu_i$ for all $1 \leq i < j$ and $\lambda_j < \mu_j$. If $\mu_j > \lambda_j + 1$ then there exists a shape that lies between λ and μ, and so μ does not cover λ. Hence $\mu_j = \lambda_j + 1$.

If $\mu_{j+1} > \lambda_{j+1}$ then λ does not cover μ (by inspection). Therefore we must have that $\mu_{j+1} = \lambda_{j+1}$ or $\mu_{j+1} = \lambda_{j+1} - 1$. Moreover, if $\mu_{j+1} = \lambda_{j+1}$ then we must have either $\mu_{j+2} = \lambda_{j+2}$ or $\mu_{j+2} = \lambda_{j+2} - 1$, and so on. Let k be the first value such that $\mu_k = \lambda_k - 1$.

If there exists $l > k$ such that $\mu_l > \lambda_l$ then, again, there will exist a shape that lies between λ and μ in the shape lattice. Hence, $\mu_l = \lambda_l$ for all $l > k$, and so the proposition holds. □

It should be noted that this condition is necessary but not sufficient. The shape $(11, 4)$ dominates the shape $(10, 4, 1)$ but, despite satisfying the conditions of Proposition 4, does not cover it as

$$(10,4,1) \trianglelefteq (11,3,1) \trianglelefteq (11,4)\,.$$

Martin and Sagan investigated the properties of orbit-λ-transitive groups in the specific case when G acts transitively, and termed such groups λ-transitive groups. In [**8**], they obtained the following generalisation of the Livingstone-Wagner theorem [**7**].

THEOREM 5. *Let μ dominate λ, and suppose that $G \le S_n$ is λ-transitive. Then G is μ-transitive.*

The following, more general, version of the Livingstone-Wagner theorem [**7**] is also true:

THEOREM 6. *Let μ dominate λ and suppose that $G \le S_n$ is orbit-λ-transitive. Then G is orbit-μ-transitive.*

Proof. It is enough to prove this in the case where μ covers λ in the shape lattice, since we can then prove the theorem by induction on the length of the chain connecting them. By Proposition 4, this means that there exist $j < k$ such that

$$\mu_j = \lambda_j + 1, \quad \mu_k = \lambda_k - 1, \quad \mu_i = \lambda_i \text{ for } i \neq j,k.$$

Suppose that G is orbit-λ-transitive. Let (S_i) and (T_i) be two partitions with the same structure that have $|S_i| = |T_i| = \mu_i$ for all i. We have to show that some element of g carries the first partition to the second. This follows from the Martin–Sagan result [**8**] if G is transitive, so we may suppose not.

Since $\mu_j > \mu_k$, there is an orbit Δ of G such that $|\Delta \cap S_j| > |\Delta \cap S_k|$. Choose $x \in \Delta \cap S_j$ and let $S_j^* = S_j \setminus \{x\}$, $S_k^* = S_k \cup \{x\}$, and $S_i^* = S_i$ for $i \neq j,k$; construct T^* similarly. Then S^* and T^* have the same structure and are partitions of shape λ, and so there exists $g \in G$ carrying S^* to T^*. This element carries $S_i \setminus \Delta$ to $T_i \setminus \Delta$ for all i, so we can assume these sets are equal.

Since $|\Delta \cap S_j| > |\Delta \cap S_k|$, the shape λ' of Δ induced by S^* is dominated by the shape μ' induced by S. Now the stabiliser of all the sets $S_i \setminus \Delta$ is transitive on partitions of Δ of shape λ' (as G acts transitively on partitions of shape λ). By the result of Martin and Sagan [**8**] again, it is transitive on partitions of shape μ', so there is an element h fixing all $S_i \setminus \Delta$ and mapping all $S_i \cap \Delta$ to $T_i \cap \Delta$. So we are finished. □

COROLLARY 7. *Let G be a permutation group acting on a finite set V. The set of normal subgroups $\Omega_\lambda(G)$ (where λ is any shape of V) form a lattice.*

Proof. Suppose λ and μ are shapes of V and that λ is dominated by μ. As $\Omega_\lambda(G)$ is orbit-λ-transitive, it is also orbit-μ-transitive. Therefore $\Omega_\lambda(G) \le \Omega_\mu(G)$ as $\Omega_\mu(G)$ is maximal. We may now conclude that the set of subgroups $\Omega_\lambda(G)$ forms a lattice because the set of shapes λ forms a lattice. □

COROLLARY 8. *If G is an orbit-t-homogeneous permutation group acting on a finite set V, where $|V| \ge 2t-1$ and $t > 1$, then G is orbit-$(t-1)$-homogeneous.*

It is clear that if $\lambda = (n)$ or $\lambda = (n-1,1)$ then $\Omega_\lambda(G) = G$.

For $t \in \{1,2,\ldots,\lfloor n/2 \rfloor\}$, let $\Omega_t(G)$ denote $\Omega_\lambda(G)$ where $\lambda = (n-t,t)$. Hence $\Omega_t(G)$ is the maximal subgroup of G that is orbit-t-homogeneous.

THEOREM 9. *Suppose G is a permutation group with degree n that acts on a set V with d orbits. If $\lambda_1 \trianglerighteq \lambda_2 \trianglerighteq \ldots \trianglerighteq \lambda_k$ is a chain of shapes of V then*

$$|\{\Omega_{\lambda_i}(G) : 1 \leq i \leq k\}| \leq d+2$$

Proof. Every shape except (n) and $(n-1,1)$ is dominated by $(n-2,2)$. Hence, by Theorem 6, $\Omega_{\lambda_i}(G)$ is orbit-2-homogeneous for all $1 \leq i \leq k$ except, possibly, when $i = 1$ and $i = 2$. Now, $\Omega_2(G)$ acts primitively on its orbits; so, for each $\lambda_i \trianglelefteq (n-2,2)$, the normal subgroup $\Omega_{\lambda_i}(G)$ must act either transitively or trivially on each $\Omega_2(G)$-orbit. Furthermore, if $\Omega_{\lambda_i}(G)$ acts trivially on a $\Omega_2(G)$-orbit then it acts trivially on all the $\Omega_2(G)$-orbits in the same G-orbit.

Therefore, either $\Omega_{\lambda_{i+1}}(G)$ acts trivially on exactly the same G-orbits as $\Omega_{\lambda_i}(G)$, or there exists at least one G-orbit on which $\Omega_{\lambda_{i+1}}(G)$ acts trivially and $\Omega_{\lambda_i}(G)$ does not. Suppose $\Omega_{\lambda_{i+1}}(G)$ acts trivially on exactly the same G-orbits as $\Omega_{\lambda_i}(G)$. Then $\Omega_{\lambda_{i+1}}(G) \leq \Omega_{\lambda_i}(G)$ (by Theorem 6) and, if P and Q are partitions of shape λ_{i+1} that have the same structure with respect to $\Omega_{\lambda_i}(G)$, then P and Q have the same structure with respect to $\Omega_{\lambda_{i+1}}(G)$. This means that there exists $\sigma \in \Omega_{\lambda_{i+1}}(G) \leq \Omega_{\lambda_i}(G)$ such that $P\sigma = Q$. Hence, $\Omega_{\lambda_i}(G)$ is orbit-λ_{i+1}-transitive, which contradicts the maximality of $\Omega_{\lambda_{i+1}}(G)$ unless $\Omega_{\lambda_i}(G) = \Omega_{\lambda_{i+1}}(G)$.

So, if $\Omega_{\lambda_{i+1}}(G)$ acts trivially on exactly the same G-orbits as $\Omega_{\lambda_i}(G)$ then $\Omega_{\lambda_{i+1}}(G) = \Omega_{\lambda_i}(G)$. Alternatively, if $\Omega_{\lambda_{i+1}}(G) < \Omega_{\lambda_i}(G)$ then there must exist at least one G-orbit on which $\Omega_{\lambda_{i+1}}(G)$ acts trivially and $\Omega_{\lambda_i}(G)$ does not. If $\Omega_{\lambda_i}(G)$ acts trivially on every G-orbit then $\Omega_{\lambda_i}(G) = 1_G$. Hence the result holds. □

This means that in the case of orbit-t-homogeneous groups things are, in fact, quite restricted.

PROPOSITION 10. *If G is transitive then one of the following holds:*
(a) $\Omega_1(G) = G$, $\Omega_t(G) = 1_G$ for all $1 < t \leq n/2$.
(b) There is a non-trivial normal subgroup $N \trianglelefteq G$ such that $\Omega_1(G) = G$, $\Omega_t(G) = N$ for all $1 < t \leq n/2$.
(c) There is a non-trivial normal subgroup $N \trianglelefteq G$ and an integer $m > 1$ such that $\Omega_1(G) = G$, $\Omega_t(G) = N$ for $1 < t \leq m$, and $\Omega_t(G) = 1_G$ for all $m < t \leq n/2$.

Proof. Set $N = \Omega_2(G)$. We note that N could equal G, 1_G or any proper normal subgroup of G. If $N = 1_G$ then case (a) applies by Theorem 6, so we assume that $N \neq 1_G$. Let t be any integer such that $2 < t \leq n/2$. $\Omega_t(G) < N$ by Theorem 6 and, as noted in the proof of Theorem 9, $\Omega_t(G)$ must act either transitively or trivially on each N-orbit. If it acts trivially on any N-orbit, then it must act trivially on the single G-orbit and so $\Omega_t(G) = 1_G$. If it acts transitively on every N-orbit, then N is orbit-t-homogeneous and $N = \Omega_t(G)$. Hence, either case (b) or case (c) must apply. □

As a series of examples, consider a group H that acts on a set of n points ($n \geq 2$), and the wreath product $G = \mathrm{Wr}(H, C_2) = (H \times H) \cdot C_2$ that acts on a set V of $2n$ points in the natural way.

(1) If $H \cong C_n$ then $\Omega_1(G) = G$ and $\Omega_t(G) = 1$ for all $1 < t \leq n$.
(2) If $H \cong S_n$ then $\Omega_1(G) = G$ and $\Omega_t(G) = H \times H$ for all $1 < t \leq n$.
(3) If H is u-homogeneous but not $(u+1)$-homogeneous, for $1 < u < n$, then $\Omega_1(G) = G$, $\Omega_t(G) = H \times H$ for $2 \leq t \leq u$, and $\Omega_t(G) = 1$ for $u < t \leq n$. Such groups exist only for $u \leq 5$ (by the main result of Livingstone and Wagner and the classification of t-transitive groups).

For intransitive groups, things are not so restricted, as the examples in the following remarks show.

Remark 11. A permutation group G with two orbits V_1 and V_2 is orbit-2-homogeneous if and only if G is 2-homogeneous on each orbit and transitive on $V_1 \times V_2$ (equivalently, the permutation characters of G on V_1 and V_2 are different). There are many examples of such groups. In particular, both, one, or neither of the actions of G on V_1 and V_2 may be faithful, as the following examples show:

(i) $\mathrm{PSL}(2, 7)$, with orbits of size 7 and 8;
(ii) $\mathrm{P\Gamma L}(2, 8)$, with orbits of size 3 and 28;
(iii) the direct product of two 2-homogeneous groups.

Remark 12. We can determine the degree of orbit-homogeneity of a permutation group all of whose orbits have size 2, as follows. Let G be such a group, with orbits $\Delta_1, \dots, \Delta_m$. To each element $g \in G$, we associate a binary m-tuple $c(g) = (c_1, \dots, c_m)$, where $c_i = 0$ if g fixes Δ_i pointwise, and $c_i = 1$ otherwise. Let $C(G) = \{c(g) : g \in G\}$. The fact that G is a group immediately implies that $C(G)$ is additively closed, that is, $C(G)$ is a linear binary code.

The *strength* of a code C of length m is the largest value t such that, given any t distinct coordinates $i_1, \dots, i_t$ and any t symbols $a_1, \dots, a_t$, there is a constant number α (depending on t but not on the coordinates or symbols) of codewords c such that $c_{i_j} = a_j$ for $j = 1, \dots, t$. It is easy to see that, if this holds for t, then it holds for any smaller natural number. In other terminology, the condition says that C is an *orthogonal array* of strength t and index α.

If C is a linear code, then the set of such codewords, if non-empty, is a coset of a subspace of C; so it is enough to require that the number of codewords is non-zero for any choice of coordinates and symbols.

Proposition 13. *Let G be a permutation group having all its m orbits of size 2. Suppose that $t \leq m$. Then G is orbit-t-homogeneous if and only if the code $C(G)$ defined above has strength at least t.*

Proof. Suppose first that $C(G)$ has strength at least t. Consider two t-sets S_1 and S_2 with the same structure. Suppose that $|S_1 \cap \Delta_i| = |S_2 \cap \Delta_i| = 1$ for $i = i_1, \dots, i_s$, where $s \leq t$. Now an element $g \in G$ maps S_1 to S_2 if and only if the corresponding codeword $c = c(g)$ has $c_{i_j} = a_j$ for $j = 1, \dots, s$, where

$$a_j = \begin{cases} 0 & \text{if } S_1 \cap \Delta_{i_j} = S_2 \cap \Delta_{i_j}, \\ 1 & \text{otherwise.} \end{cases}$$

Such an element exists by the case assumption.

Conversely, suppose that G is orbit-t-homogeneous, and let coordinates $i_1, \dots, i_t$

and symbols $a_1, \ldots, a_t \in \{0, 1\}$ be given. Choose t-sets S_1, S_2 containing one point from each orbit Δ_{i_j} for $j = 1, \ldots, t$, where the chosen points of O_{i_j} are the same if $a_j = 0$ and different if $a_j = 1$. By orbit-t-homogeneity, the set of elements g which carry S_1 to S_2 is non-empty and so is a coset of a subgroup of G; so its size is independent of the choice of i_j and a_j. These elements satisfy $c_{i_j} = a_j$ for $j = 1, \ldots, t$, where $c = c(g)$; and so $C(G)$ has strength at least t.

□

According to a theorem of Delsarte [**2**], the strength of a linear code C is one less than the minimum weight of the dual code $C^\perp$. So it is easy to construct permutation groups with any desired degree of orbit-homogeneity, from linear codes with sufficiently large minimum weight.

REMARK 14. If G has all orbits of size 3, then G is orbit-t-homogeneous if and only if its (normal) Sylow 3-subgroup is. The criterion for this is almost identical to that in Remark 12, in terms of ternary codes. Also, if G has all orbits of size 2 or 3, then G is orbit t-homogeneous if and only if the groups induced on the union of orbits of each size are. We do not give details.

REMARK 15. The situation for orbits of size 4 or more is a bit more complicated. We can give a partial description of the orbit-4-homogeneous groups as follows.

PROPOSITION 16. *Let G be orbit-4-homogeneous of degree at least 8, and let H be the third derived group of G. Then H is a direct product of simple groups taken from the list A_n $(n \geq 5)$, M_n $(n = 11, 12, 23, 24)$, and $\mathrm{PSL}(2, q)$ $(q = 5, 8, 32)$, each factor acting transitively on one G-orbit and fixing all the others pointwise.*

Proof. The 4-homogeneous groups which are not 4-transitive have been classified by Kantor [**5**], and the list of 4-transitive groups follows from the classification of finite simple groups. All of them have simple derived groups in the list in the proposition. Groups of degree at most 4 have derived length at most 3. By inspection, a group on the above list cannot act non-trivially on two different orbits in an orbit-4-homogeneous group. □

There remains some subtlety in the structure of G. For example:

- The Proposition gives no information about orbits of size at most 4. In particular, the examples described in Remarks 2 and 3 are completely invisible from this point of view.
- Any group G lying between a direct product $\prod_{i=1}^{r} S_{n_i}$ of symmetric groups and its derived group $\prod_{i=1}^{r} A_{n_i}$ (with $n_i \geq 5$ for all i) is orbit-4-homogeneous. We can add orbits of length 2 on which $G/\prod A_{n_i}$ acts as in Remark 1.
- The group $\mathrm{P\Gamma L}(2, 8)$, acting with orbits of size 3 and 9, is orbit-4-homogeneous. The transitivity on 4-sets containing one point from the orbit of length 3 follows from the 3-homogeneity of $\mathrm{PSL}(2, 8)$.

REMARK 17. The above Proposition fails for orbit-3-homogeneous groups. The groups S_6 (with two inequivalent orbits of size 6) and M_{12} (with two inequivalent orbits of size 12) are orbit-3-homogeneous but not orbit-4-homogeneous. Other examples include $(C_2^r)^m \cdot \mathrm{GL}(r, 2)$, for $m, r \geq 2$, with m orbits of size 2^r.

REMARK 18. Let G_s be s-homogeneous but not $(s+1)$-homogeneous on V_s for $1 \le s \le 5$; let $G = G_1 \times \ldots \times G_5$, acting on $V = V_1 \cup \ldots \cup V_5$. Then

$$\Omega_t(G) = \Omega_t(G_1) \times \prod_{i=t}^{5} G_i\ .$$

References

1. P. J. CAMERON, *Permutation Groups* (Cambridge Univ. Press, Cambridge, 1999).
2. P. DELSARTE, 'Four fundamental parameters of a code and their combinatorial significance', *Information and Control* 23 (1973) 407–438.
3. A. W. DENT, 'On the theory of point weight designs', doctoral thesis, University of London, 2002.
4. D. GORENSTEIN, *Finite Simple Groups: An Introduction to their Classification* (Plenum Press, New York, 1982).
5. W. M. KANTOR, '4-homogeneous groups', *Math. Z.* 103 (1968) 67–68.
6. W. M. KANTOR, 'k-homogeneous groups', *Math. Z.* 124 (1972) 261–265.
7. D. LIVINGSTONE AND A. WAGNER, 'Transitivity of finite permutation groups on unordered sets', *Math. Z.* 90 (1965) 393-403.
8. W. J. MARTIN AND B. E. SAGAN, 'A new notion of transitivity for groups and sets of permutations', to appear in *The Journal of the London Mathematical Society.*
9. G. DE B. ROBINSON, *Representation Theory of the Symmetric Group* (Edinburgh University Press, Edinburgh, 1961).

Peter J. Cameron
School of Mathematical Sciences
Queen Mary, University of London
Mile End Road
London E1 4NS

P.J.Cameron@qmul.ac.uk

Alexander W. Dent
Information Security Group
Royal Holloway, University of London
Egham Hill
Egham, Surrey TW20 0EX

A.Dent@rhul.ac.uk